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COMMENT

Note on collective variable theory of nonlinear Schrödinger solitons

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Abstract. Inconsistencies arise in a recently developed collective variable theory of nonlinear Schrödinger solitons, as a result of a particular formulation of the energy-conservation principle in terms of the time derivative of the phase of the original field. We show that the inconsistencies are resolved either by correctly reformulating the energy-conservation principle or by directly averaging the nonlinear Schrödinger equation.

A well known example of an equation which admits pulse-like soliton solutions is the nonlinear Schrödinger equation [1]. A characteristic feature of these bright solitons is that they are localized variations of the associated field. This behaviour, which is quite similar to that of particle-like objects, has therefore naturally led to the formulation of collective-variable theories in which solitons are treated as particles [2–4]. In the standard collective-variable (CV) approach, one introduces particle-like parameters such as the centre of mass of the pulse, or the soliton width, and one considers them as dynamical variables.

In this brief comment we point out some inconsistencies in a recently developed CV theory [4], which result from a particular formulation of the energy-conservation principle in terms of the time derivative of the phase of the wavepacket associated with the soliton. We show that the inconsistencies are resolved by simply reformulating the energy-conservation principle within the average Lagrangian formalism. We first recall below the essential steps of the derivation of the CV equations of motion by the approach developed in [4], in order to give the reader a deep insight into the statement of the problem. The nonlinear Schrödinger equation (NLSE) may take the following general form:

$$\mathrm{i}\frac{\partial\psi}{\partial t} + \frac{1}{2}\frac{\partial^2\psi}{\partial x^2} + \alpha^2|\psi|^2\psi = 0 \tag{1}$$

where α^2 designates the nonlinear coefficient of the medium. Here t and x represent the time and space coordinates, respectively. In [4], the first step of the derivation of the CV equations is to decompose the original field in the following way:

$$\psi(t,x) = \phi(t,x)\exp[\mathrm{i}S(t,x)] \tag{2}$$

where ϕ and S represent, respectively, the amplitude and phase of the original field ψ. Next, the ansatz for ϕ is chosen as follows [4]:

$$\phi(t,x) = \left[\frac{1}{\sigma(t)}\right]^{1/2}\tilde{\phi}[\sigma(t)^{-1}(x - x_c(t))] \tag{3}$$

where x_c and σ represent, respectively, the centre of mass and the width of the wavepacket. Then one integrates the imaginary part of the NLSE (1) to obtain the following expression for the phase of the original field [4]:

$$S(t,x) = B(t) + \dot{x}_c(t)[x - x_c(t)] + \frac{1}{2}\frac{\dot{\sigma}(t)}{\sigma(t)}[x - x_c(t)]^2 \tag{4}$$

where $\dot{\sigma} = \mathrm{d}\sigma/\mathrm{d}t$, $\dot{x_c} = \mathrm{d}x_c/\mathrm{d}t$ and $B(t)$ is a constant of integration.

We would now like to point out the following fundamental point. In [4], the equation of motion for the pulse width σ was obtained by using an energy-conservation principle, that is, by setting the time derivative of the energy to zero. However, in [4], the energy was chosen to be of the form

$$E = -\left\langle \frac{\partial S}{\partial t} \right\rangle = \dot{\sigma}^2 \frac{\gamma^2}{2} + \frac{\dot{x}_c^2}{2} - \alpha^2 \frac{\delta^2}{\sigma} + \frac{1}{2}\frac{\mu^2}{\sigma^2} \tag{5}$$

where the averaging is the usual quantum expectation value $\langle O \rangle = \int \psi^* O \psi \, \mathrm{d}x$. In other words, taking the time derivative of equation (5) leads to the following equation for σ:

$$\ddot{\sigma} = \frac{1}{\sigma^3}\frac{\mu^2}{\gamma^2} - \frac{\alpha^2}{\sigma^2}\frac{\delta^2}{\gamma^2} \tag{6}$$

where $\gamma^2 = \int_{-\infty}^{+\infty} \xi^2 \tilde{\phi}^2 \, \mathrm{d}\xi$, $\mu^2 = \int_{-\infty}^{+\infty} \tilde{\phi}_\xi^2 \, \mathrm{d}\xi$, $\delta^2 = \int_{-\infty}^{+\infty} \tilde{\phi}^4 \, \mathrm{d}\xi$, with $\xi \equiv (x - x_c)/\sigma$. Equation (6) corresponds exactly to equation (26) of [4]. We show below that the averaged Lagrangian approach leads to an equation of motion for σ that differs from equation (6), and we resolve this contradiction.

In this context it is worth recalling that the NLSE (1) can be restated as a variational problem corresponding to the Lagrangian density given by

$$L = -\frac{1}{2}\left|\frac{\partial \psi}{\partial x}\right|^2 + \frac{\alpha^2}{2}|\psi|^4 + \frac{\mathrm{i}}{2}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) \tag{7}$$

where ψ^* designates the complex conjugate of ψ. In other words, the NLSE (1) corresponds exactly to the following variational equation:

$$\frac{\delta L}{\delta \psi} = \frac{\partial}{\partial t}\left[\frac{\partial L}{\partial[\partial \psi^*/\partial t]}\right] + \frac{\partial}{\partial x}\left[\frac{\partial L}{\partial[\partial \psi^*/\partial t]}\right] - \frac{\partial L}{\partial \psi^*} = 0 \tag{8}$$

which results from the variational principle [2, 3] $\delta \iint L(\psi, \psi^*, \psi_t, \psi_t^*, \psi_x, \psi_x^*) \, \mathrm{d}x \, \mathrm{d}t = 0$, where $\psi_t = \partial\psi/\partial t$ and $\psi_x = \partial\psi/\partial x$. Thus, the above Lagrangian which, is consistent with the original field equation (1), leads to the following Hamiltonian:

$$H = -\int L \, \mathrm{d}x + \int \left[\frac{\mathrm{i}}{2}\left(\psi \frac{\partial \psi^*}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)\right] \mathrm{d}x = \int \left[\tfrac{1}{2}|\psi_x|^2 - \tfrac{1}{2}\alpha^2|\psi|^4\right] \mathrm{d}x. \tag{9}$$

Then, using equations (2)–(4), we obtain the following expression:

$$H = \dot{\sigma}^2 \frac{\gamma^2}{2} + \frac{\dot{x}_c^2}{2} - \alpha^2 \frac{\delta^2}{2\sigma} + \frac{1}{2}\frac{\mu^2}{\sigma^2} \tag{10}$$

which is consistent with the original field equations and which differs from $E = -\langle \partial S/\partial t \rangle$ [4] by

$$E - H = -\alpha^2 \frac{\delta^2}{2\sigma}. \tag{11}$$

Taking the time derivative of H in equation (10) yields the following equation:

$$\ddot{\sigma} = \frac{1}{\sigma^3}\frac{\mu^2}{\gamma^2} - \frac{\alpha^2}{2\sigma^2}\frac{\delta^2}{\gamma^2} \tag{12}$$

which is consistent with the original NLSE (1). Therefore the fundamental point here is that equation (12) differs from that found in the previous work [4] (see equation (6)), by the presence of the factor $\frac{1}{2}$ in the last term of the right-hand side of equation (12). We attribute the disagreement between equations (12) and (6) to the particular formulation of the energy as $E = -\langle \dot{S} \rangle$ which comes directly from quantum mechanical principles based on linear field theories, and therefore cannot be straightforwardly extended to nonlinear field theories.

As a final remark, note that in the case of the fundamental soliton ($\alpha^2 = 1$), ϕ is a sech function of x_c and σ, which here are both constant, and so one can take $x_c = 0$. In this case the ansatz function ϕ takes the form $\phi = \tilde{\phi} = \text{sech}(x)$, with $\sigma = 1$, and all the other soliton parameters can be calculated explicitly: $\mu^2 = \int_{-\infty}^{+\infty} \phi_x^2 \, \mathrm{d}x = \frac{2}{3}$ and $\delta^2 = \int_{-\infty}^{+\infty} \phi^4 \, \mathrm{d}x = \frac{4}{3}$. Then equation (12) reduces to $\ddot{\sigma} = 0$. In contrast, equation (6) does not fulfil this consistency with the fundamental soliton problem. In conclusion, we would like to emphasize that the soliton parameters are quantities that play a crucial role in many practical applications such as in signal processing and communications [2]. Extreme care should therefore be taken when defining those quantities in soliton-bearing systems.

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